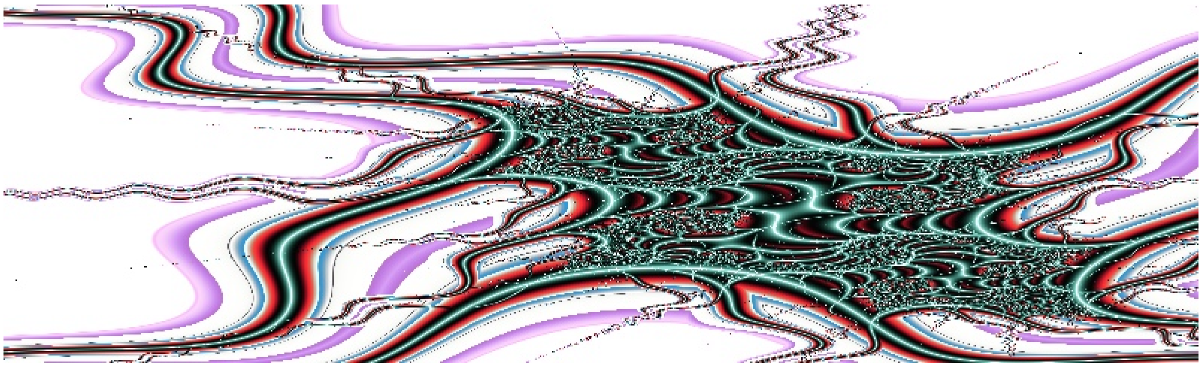

Classroom Chaos 8

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0. Circle Maps

A general circle map is defined by

$$\theta_{t+1} = F(\theta_t) \bmod 1$$

In this map, θ can be considered to be the position along a circle of circumference = 1.

Question 1: If the function $F(\theta)$ is linear, does the map generate cyclic orbits for any starting value of θ in [0,1]?

Test this assertion by assuming $F(\theta) = \theta + \Omega$, where Ω is a constant in [0,1], and creating a spreadsheet that calculates the sequence generated by the map. Use a designated cell at the top of your sheet for changing the value of Ω. For different values of Ω, determine how many iterations are required to return to the starting point, and how many rotations around the circle the point makes. Define the *winding number*, W, as

$$W = \frac{\#\text{ of times around circle}}{\#\text{ of iterations}}$$

How is W related to Ω? Will all values of Ω and W generate cyclic orbits?

Question #2: Use a straight-edge to create a stair-step diagram for the map

$$\theta_{t+1} = \theta_t + 0.4 \quad \bmod 1$$

How does your stair-step diagram coincide with your value of winding number for this map?

Question #3: What does your stair-step diagram look like for the map

$$\theta_{t+1} = \theta_t + (\sqrt{2} - 1.014) \quad \bmod 1\text{?}$$

<u>Question #4</u>: If the mod 1 were removed from your map in Question #3, show that defining the Winding # as

$$W = \lim_{t \to \infty} \frac{\theta_t - \theta_0}{t}$$

yields the same values for W that you obtained in Question 1, and provides a reasonable way of defining W for maps that do not generate cyclic orbits.

1. Chicken Hearts

The chicken heart data of Glass, Goldberger, Courtemanche, and Shrier can be simulated using the non-linear circle map

$$\theta_{t+1} = \theta_t + \Omega - \left(\frac{K}{2\pi}\right) \sin(2\pi\theta_t) \quad \text{mod}\,1$$

where $0 \le K \le 1$, and K measures the non-linearity of the map.

The Devils' Staircase: Setting K=1, create a spreadsheet that implements this map (sans the mod 1 piece) for 100 values of Ω, where the start and end values of Ω can be changed easily. (You will need 100 columns for this sheet. Let your map iterate 250 times for each value of Ω.) For each value of Ω, numerically calculate the approximate winding # W.

Plot W vs. Ω on an *x*-*y* scatter graph, where $0 \le \Omega \le 1$. Describe the significance of the W values corresponding to the steps that you see.

Zoom in on different parts of your staircase by changing the starting and ending values of Ω. How many steps are in your staircase for $0 \le \Omega \le 1$?

<u>Stair-steps from the Devils' Staircase</u> Create a stair-step diagram for your map (with the mod 1 piece, of course) for a value of Ω that yields a winding # plateau. Does your diagram produce the expected result?